Electronic Supplementary Material for

# Coevolution of third-party punishment and punishment reputation dependency

Taisho Ozaki and Yasuo Ihara

Letters on Evolutionary Behavioral Science
Vol.16 No.2 (2025) 36-41 doi: 10.5178/lebs.2025.129

## Appendix A. Model 1

*Expected total payoffs*

Let $D_N(k)$, $D_P(k)$, and $D_e(k)$ denote the expected payoff of IS, ISp, and ALLD, respectively, from the $k$th round of a game. We have

$$x_N D_N(k) = x_{N0} P_{N0} + x_{N1} P_{N1}, \tag{A1a}$$

$$x_P D_P(k) = x_{P0} P_{P0} + x_{P1} P_{P1}, \tag{A1b}$$

$$y D_e(k) = y_0 Q_0 + y_1 Q_1, \tag{A1c}$$

where $P_{N0}$, $P_{N1}$, $P_{P0}$, $P_{P1}$, $Q_0$, and $Q_1$ represent the expected payoffs of Good IS, Bad IS, Good ISp, Bad ISp, Good ALLD, and Bad ALLD, respectively, as shown in Table 1. Thus, the expected total payoffs of IS, ISp, and ALLD are given respectively by

$$D_N = \sum_{k=1}^{\infty} w^{k-1} D_N(k), \tag{A2a}$$

$$D_P = \sum_{k=1}^{\infty} w^{k-1} D_P(k), \tag{A2b}$$

$$D_e = \sum_{k=1}^{\infty} w^{k-1} D_e(k), \tag{A2c}$$

where $w$ is the probability that another round of game is played, given the current one.

*Competition between IS and ALLD*

In the absence of ISp ($x_{P1} = 0$), (1) reduces to

$$x_{N1}(k+1) = \frac{2 + (1-e)x}{3} x_{N1}(k) + \frac{(1-e)x}{3} y_1(k), \tag{A3a}$$

$$y_1(k+1) = \frac{2}{3} y_1(k), \tag{A3b}$$

where $x_{N1}(k)$ and $y_1(k)$ denote the frequencies of Good IS and Good ALLD in the $k$th round, respectively. Using the assumptions $x_{N1}(1) = x$ and $y_1(1) = 1-x$, we obtain

$$x_{N1}(k) = \left[\frac{2 + (1-e)x}{3}\right]^{k-1} - (1-x)\left(\frac{2}{3}\right)^{k-1}, \qquad \text{(A4a)}$$

$$y_1(k) = (1-x)\left(\frac{2}{3}\right)^{k-1}. \qquad \text{(A4b)}$$

From Table 1, (A1), and (A4), we derive

$$D_N(k) = \frac{1-e}{3}\left\{(b-c)\left[\frac{2+(1-e)x}{3}\right]^{k-1} - b(1-x)\left(\frac{2}{3}\right)^{k-1}\right\}, \qquad \text{(A5a)}$$

$$D_e(k) = \frac{(1-e)bx}{3}\left(\frac{2}{3}\right)^{k-1}. \qquad \text{(A5b)}$$

Thus, from (A2), we have

$$D_N = (1-e)\left\{\frac{b-c}{3-[2+(1-e)x]w} - \frac{b(1-x)}{3-2w}\right\}, \qquad \text{(A6a)}$$

$$D_e = \frac{(1-e)bx}{3-2w}. \qquad \text{(A6b)}$$

Hence, we show that $D_N > D_e$ holds if $x > x_N^*$, where

$$x_N^* = \frac{(3-2w)c}{(1-e)wb}, \qquad \text{(A7)}$$

and $D_N < D_e$ if $x < x_N^*$. While $x_N^* > 0$ is always true, $x_N^* < 1$ holds if and only if

$$\frac{w}{3-2w} > \frac{c}{(1-e)b}. \qquad \text{(A8)}$$

Note that $b/c > 1/(1-e)$ is necessary for (A8), and thus $x > x_N^*$. Therefore, in the absence of ISp, rare IS can never invade a population of ALLD, and a population of IS resists ALLD's invasion if and only if (A8) is satisfied.

*Competition between IS and ISp*

In the absence of ALLD ($y = 0$), (1) reduces to

$$x_{N1}(k+1) = \frac{2}{3}x_{N1}(k) + \frac{(1-e)x_N}{3}\big(x_{N1}(k) + x_{P1}(k)\big), \qquad \text{(A9a)}$$

$$x_{P1}(k+1) = \frac{2}{3}x_{P1}(k) + \frac{1 - x_N - ex_{P1}(k)}{3}\big(x_{N1}(k) + x_{P1}(k)\big). \qquad \text{(A9b)}$$

Let us first assume $e = 0$. Then (A9) becomes

$$x_{N1}(k+1) = \frac{2+x_N}{3}x_{N1}(k) + \frac{x_N}{3}x_{P1}(k), \qquad \text{(A10a)}$$

$$x_{P1}(k+1) = \frac{1-x_N}{3} x_{N1}(k) + \frac{3-x_N}{3} x_{P1}(k). \quad \text{(A10b)}$$

Solving (A10) for $x_{N1}(1) = x_N$ and $x_{P1}(1) = 1-x_N$, we obtain

$$x_{N1}(k) = x_N, \quad \text{(A11a)}$$

$$x_{P1}(k) = 1 - x_N. \quad \text{(A11b)}$$

Thus, we have

$$D_N(k) = D_P(k) = \frac{b-c}{3}, \quad \text{(A12)}$$

which means that $D_N = D_P$ always holds. Therefore, in the absence of ALLD and without inadvertent failure to provide help, IS and ISp are selectively neutral.

As for the case of $e > 0$, from (A9), we have

$$x_{N1}(k+1) + x_{P1}(k+1) = \left(x_{N1}(k) + x_{P1}(k)\right)\left(1 - \frac{e\left(1 - x_{P0}(k)\right)}{3}\right). \quad \text{(A13)}$$

Suppose that $x_{P0}(k)$ is negligibly small, in which case (A13) is approximated by

$$x_{N1}(k+1) + x_{P1}(k+1) \approx \left(x_{N1}(k) + x_{P1}(k)\right)\left(1 - \frac{e}{3}\right). \quad \text{(A14)}$$

Solving (A14) for $x_{N1}(1) = x_N$ and $x_{P1}(1) = 1-x_N$, we have

$$x_{N1}(k) + x_{P1}(k) \approx \left(1 - \frac{e}{3}\right)^{k-1}. \quad \text{(A15)}$$

Given (A15), we derive

$$x_{N1}(k) \approx x_N \left(1 - \frac{e}{3}\right)^{k-1}, \quad \text{(A16a)}$$

$$x_{P1}(k) \approx (1 - x_N)\left(1 - \phi(k)\right), \quad \text{(A16b)}$$

where

$$\phi(k) = \sum_{m=1}^{k-1}\left(\left[\frac{1}{3} - \frac{1-e}{3}\left(1 - \frac{e}{3}\right)^{m-1}\right]\prod_{n=m+1}^{k-1}\left[\frac{2}{3} - \frac{e}{3}\left(1 - \frac{e}{3}\right)^{n-1}\right]\right). \quad \text{(A16c)}$$

Using (A15) again, we have

$$D_P(k) - D_N(k) \approx \frac{1}{3}\left\{(1-e)b\left[1 - \phi(k) - \left(1 - \frac{e}{3}\right)^{k-1}\right] - e\alpha\left(1 - \frac{e}{3}\right)^{k-1}\right\}, \quad \text{(A17)}$$

from which we derive

$$3(D_P - D_N)$$

$$\approx (1-e)b\left[\sum_{k=1}^{\infty}\left(1 - \phi(k)\right)w^{k-1} - \frac{1}{1 - w\left(1 - \frac{e}{3}\right)}\right] - \frac{e\alpha}{1 - w\left(1 - \frac{e}{3}\right)}. \quad \text{(A18)}$$

Therefore, under the assumption that $x_{P0}(k)$ is negligibly small, $D_P > D_N$ holds if $\alpha < \alpha^*$, where

$$\alpha^* = \frac{(1-e)b}{e}\left\{\left[1 - w\left(1 - \frac{e}{3}\right)\right]\sum_{k=1}^{\infty}(1-\phi(k))w^{k-1} - 1\right\}, \tag{A19}$$

and $D_P < D_N$ if $\alpha > \alpha^*$. Figure A1 compares $\alpha^*$ with the threshold cost of punishment obtained by numerical iterations of (A9). For the parameter values examined, the approximation is quite good.

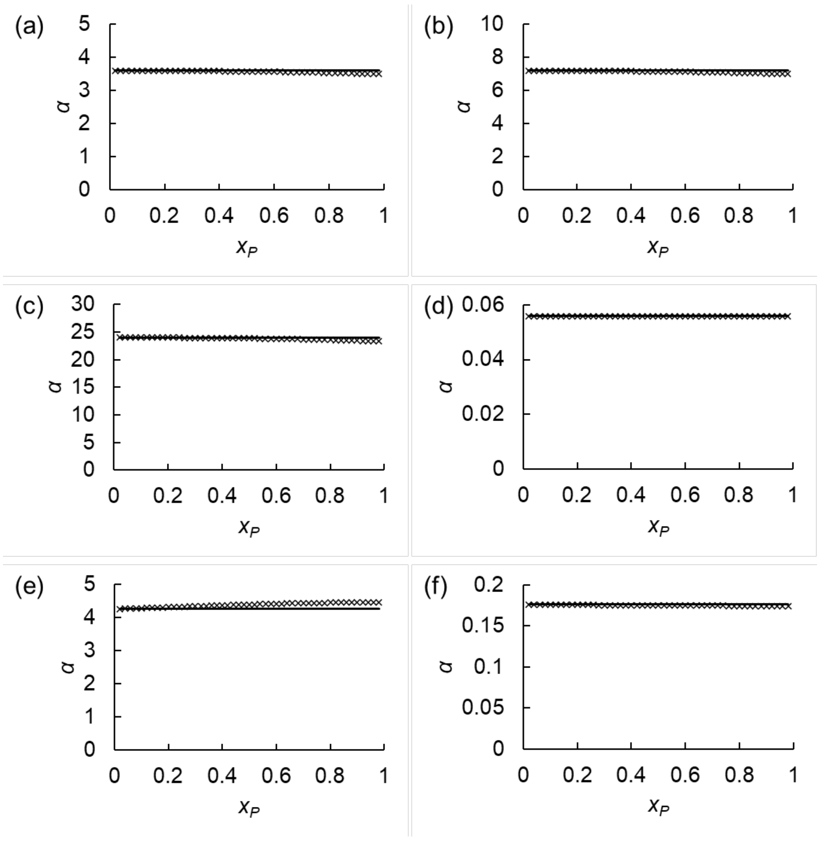

**Figure A1.** The approximated (solid lines) and numerically calculated (crosses) threshold cost of punishment, $\alpha^*$, below which $D_P > D_N$ holds, across $x_P$ in the absence of ALLD in Model 1. The approximated values are based on (A19), and the numerically calculated values are based on iterations of (A9) for 400 steps. Parameter values used are $c = 10$, $\beta = 20$ (a) $e = 0.05$, $w = 0.975$, $b = 15$, (b) $e = 0.05$, $w = 0.975$, $b = 30$, (c) $e = 0.05$, $w = 0.975$, $b = 100$, (d) $e = 0.05$, $w = 0.5$, $b = 15$, (e) $e = 0.1$, $w = 0.975$, $b = 15$, (f) $e = 0.01$, $w = 0.975$, $b = 15$.

*Competition between ISp and ALLD*

In the absence of IS ($x_N = 0$),

$$x_{P1}(k+1) = \frac{2}{3}x_{P1}(k) + \frac{(1-e)x}{3}\big(x_{P1}(k) + y_1(k)\big)$$
$$+\frac{1-(1-e)x}{3}x_{P0}(k)\big(x_{P1}(k) + y_1(k)\big), \qquad \text{(A20a)}$$

$$y_1(k+1) = \frac{2}{3}y_1(k). \qquad \text{(A20b)}$$

Suppose that $x_{P0}(k)$ is negligibly small, so that (A20a) is approximated by

$$x_{P1}(k+1) \approx \frac{2+(1-e)x}{3}x_{P1}(k) + \frac{(1-e)x}{3}y_1(k). \qquad \text{(A21)}$$

Given (A21), we obtain

$$x_{P1}(k) \approx \left[\frac{2+(1-e)x}{3}\right]^{k-1} - (1-x)\left(\frac{2}{3}\right)^{k-1}, \qquad \text{(A22a)}$$

$$y_1(k) = (1-x)\left(\frac{2}{3}\right)^{k-1}. \qquad \text{(A22b)}$$

From (A22), we derive

$$D_P(k) - D_e(k)$$
$$\approx \frac{(1-e)(b-c) - \alpha + (1-e)(\alpha+\beta)x}{3}\left[\frac{2+(1-e)x}{3}\right]^{k-1}$$
$$-\frac{(1-e)b}{3}\left(\frac{2}{3}\right)^{k-1}. \qquad \text{(A23)}$$

Hence,

$$D_P - D_e \approx \frac{(1-e)(b-c) - \alpha + (1-e)(\alpha+\beta)x}{3-[2+(1-e)x]w} - \frac{(1-e)b}{3-2w}. \qquad \text{(A24)}$$

Thus, under the assumption that $x_{P0}(k)$ is negligible, we show that $D_P > D_e$ holds if $x > x_P^*$, where

$$x_P^* = \frac{(3-2w)[(1-e)c+\alpha]}{(1-e)[(1-e)wb + (\alpha+\beta)(3-2w)]}, \qquad \text{(A25)}$$

and $D_P < D_e$ if $x < x_P^*$. Equation (A25) shows that $x_P^* > 0$ is always true and that $x_P^* < 1$ holds if and only if

$$\frac{w}{3-2w} > \frac{(1-e)c + e\alpha - (1-e)\beta}{(1-e)^2 b}. \qquad \text{(A26)}$$

This suggests that ISp never invades a population of ALLD, and that ALLD does not invade a population of ISp when (A26) is satisfied. Note that (A26) is always true if $\beta > c+[e/(1-e)]\alpha$. It is also shown that $x_P^* < x_N^*$ holds, in which case the threshold frequency of ISp above which it is selectively favored over ALLD is lower than that of IS, if and only if

$$\frac{w}{3-2w} < \frac{c}{b}\left(1+\frac{\beta}{\alpha}\right). \tag{A27}$$

Figure A2 shows approximated and numerically calculated values of $D_P\text{–}D_e$ across $x_P$. Note that $D_P\text{–}D_e = 0$ at $x_P = x_P^*$. For the parameter values examined, the approximation by (A25) is reasonably good, although it may slightly overestimate $x_P^*$ when $w$ is large.

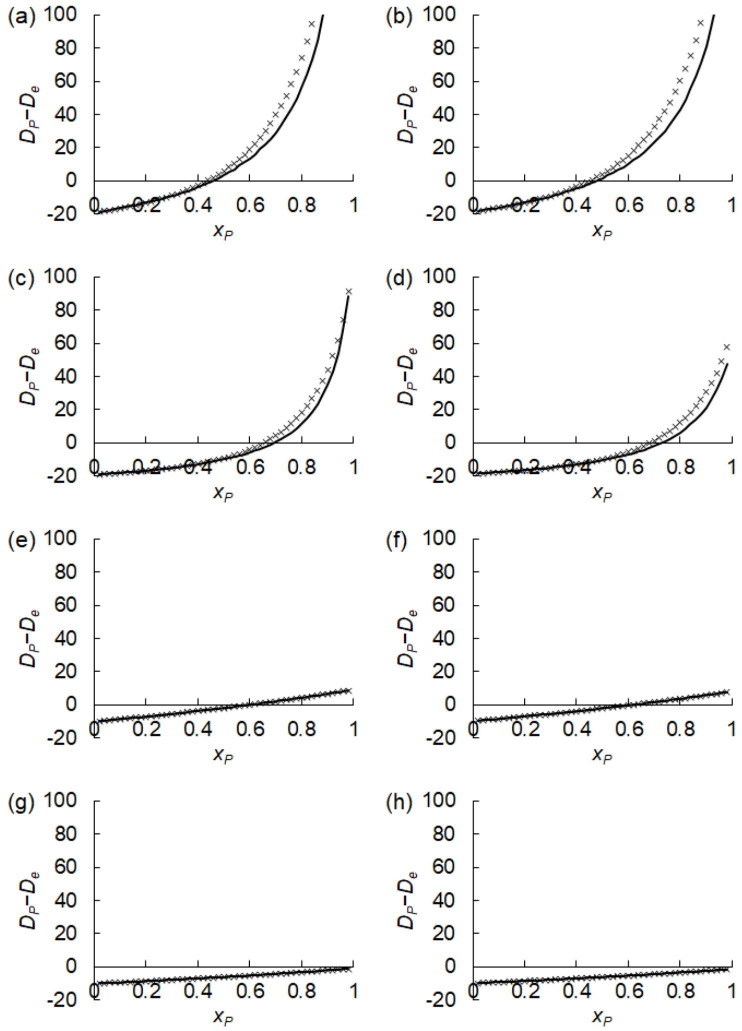


**Figure A2.** The approximated (solid lines) and numerically calculated (crosses) relative advantage of ISp over ALLD ($D_P\text{–}D_e$) across $x_P$ in the absence of IS in Model 1. The approximated values are based on (A17), and the numerically calculated values are based on iterations of (A13) for 400 steps. Parameter values used are $b = 15$, $c = 10$, $\alpha = 10$, (a) $e = 0$, $w = 0.975$, $\beta = 20$, (b) $e = 0.05$, $w = 0.975$, $\beta = 20$, (c) $e = 0$, $w = 0.975$, $\beta = 5$, (d) $e = 0.05$, $w = 0.975$, $\beta = 5$, (e) $e = 0$, $w = 0.5$, $\beta = 20$, (f) $e = 0.05$, $w = 0.5$, $\beta = 20$, (g) $e = 0$, $w = 0.5$, $\beta = 5$, and (h) $e = 0.05$, $w = 0.5$, $\beta = 5$.

*The three-strategy dynamics*

Figure A3 shows sample trajectories of the frequencies of IS, ISp, and ALLD in Model 1. See Figure 1 for the case of $e = 0$.

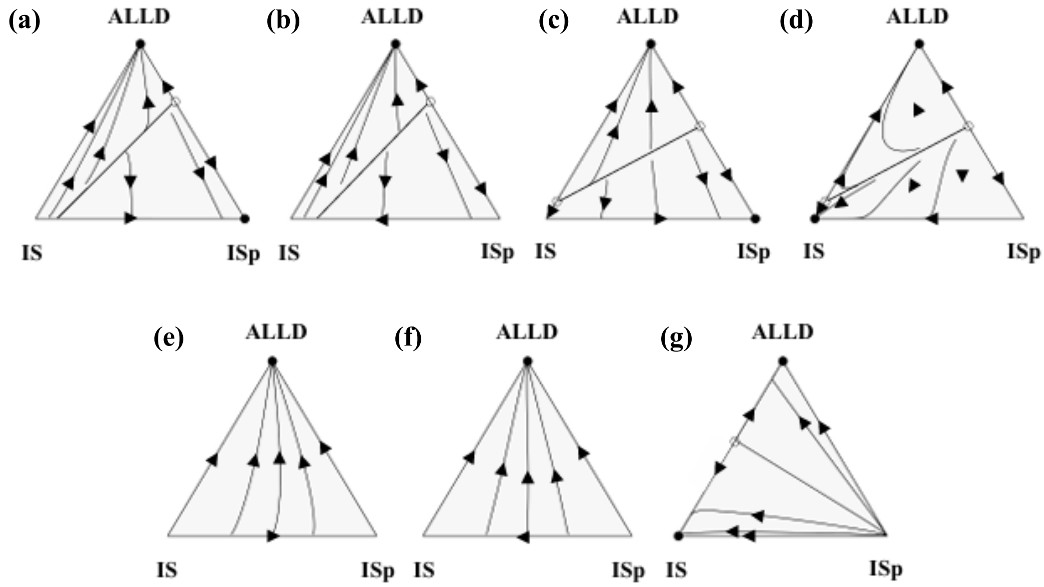


**Figure A3.** Sample trajectories of the evolutionary dynamics described by (2). (a, b) Inequality (A8) is not satisfied and (A26) is satisfied, (c, d) All (A8), (A26), and (A27) are satisfied, (e, f) Neither (A8) nor (A26) is satisfied, (g) (A8) is satisfied and (A26) is not satisfied, (a, c, e) $\alpha < \alpha^*$, and (b, d, f, g) $\alpha > \alpha^*$. Parameter values used are $e = 0.05$, $c = 10$, $w = 0.9$ (a) $b = 11$, $\alpha = 0.1$, $\beta = 20$, (b) $b = 11$, $\alpha = 1$, $\beta = 20$, (c) $b = 15$, $\alpha = 1$, $\beta = 10$, (d) $b = 15$, $\alpha = 10$, $\beta = 20$, (e) $b = 11$, $\alpha = 0.1$, $\beta = 1$, (f) $b = 11$, $\alpha = 1$, $\beta = 1$, and (g) $b = 30$, $\alpha = 500$, $\beta = 1$.

## Appendix B. Model 2

*Sensitivity analysis*

First, we investigated how our results may be affected by changing values of the cost of punishing non-helping donors, $\alpha$, and the increase of a punisher's punishment reputation owing to a single act of punishment, $u_p$. Figure A4 suggests, not surprisingly, that the number of generations in which the population is in the cooperative state is larger when $\alpha$ is smaller and/or $u_p$ is larger.

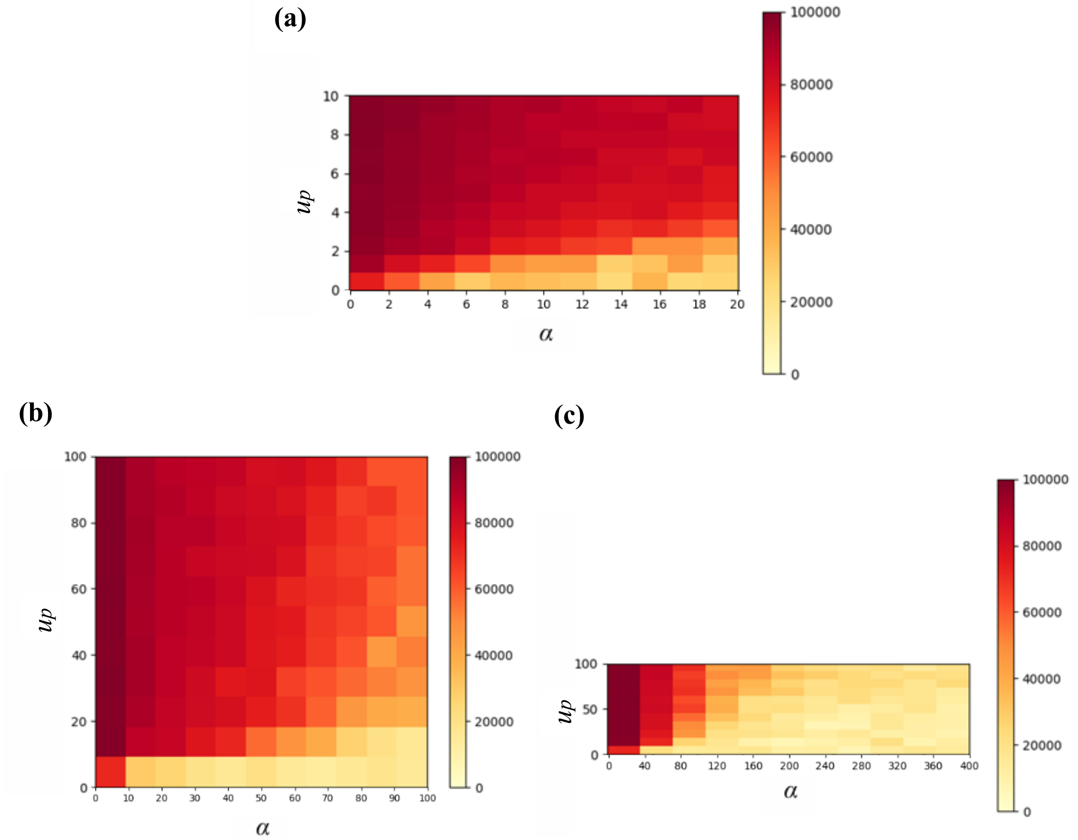


**Figure A4.** The number of generations in the cooperative state for different combinations of $\alpha$ and $u_p$. The color of each cell represents the number of generations in which the population was in the cooperative state. Parameter values used are the same as in Figure 3.

Second, we examined the relationship between the cost of being punished, $\beta$, and the number of generations in the cooperative state. As suggested by Figure A5, the number of cooperative generations increases with $\beta$.

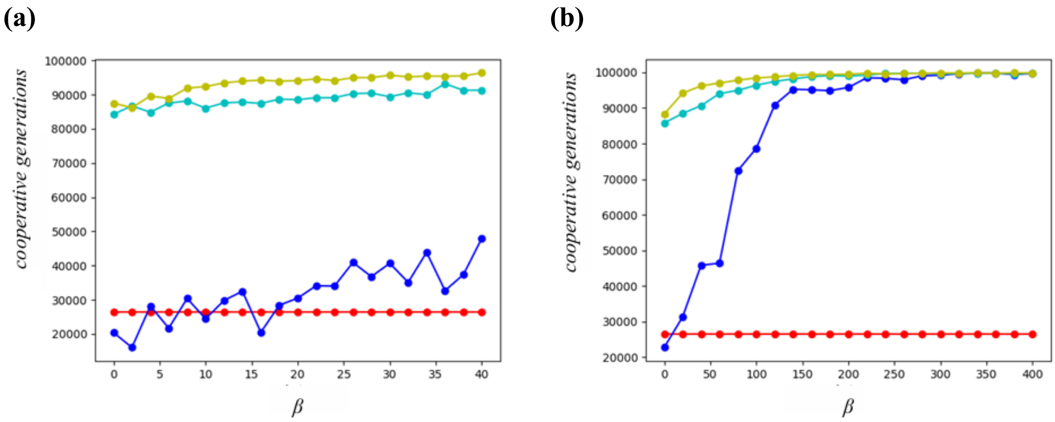


**Figure A5.** The number of generations in which the population was in the cooperative state across varying $\beta$. Different colors represents the results when all individuals were forced to be non-punishing and without the PRD (red), when the punishing trait was allowed to evolve under the condition that no individuals have the PRD (dark blue), when both punishing and PRD traits were allowed to evolve (light blue), and when the punishing trait was allowed to evolve under the condition that all individuals have the PRD (yellow). Parameter values used are the same as Figure 3.

Third, Figure A6 shows the relationship between the number of games per generation, $m$, and the number of generations in which the population is in the cooperative state. It suggests that the number of cooperative generations increases with $m$. In addition, it is also suggested that the punishing and PRD traits facilitate cooperation most significantly for intermediate $m$.

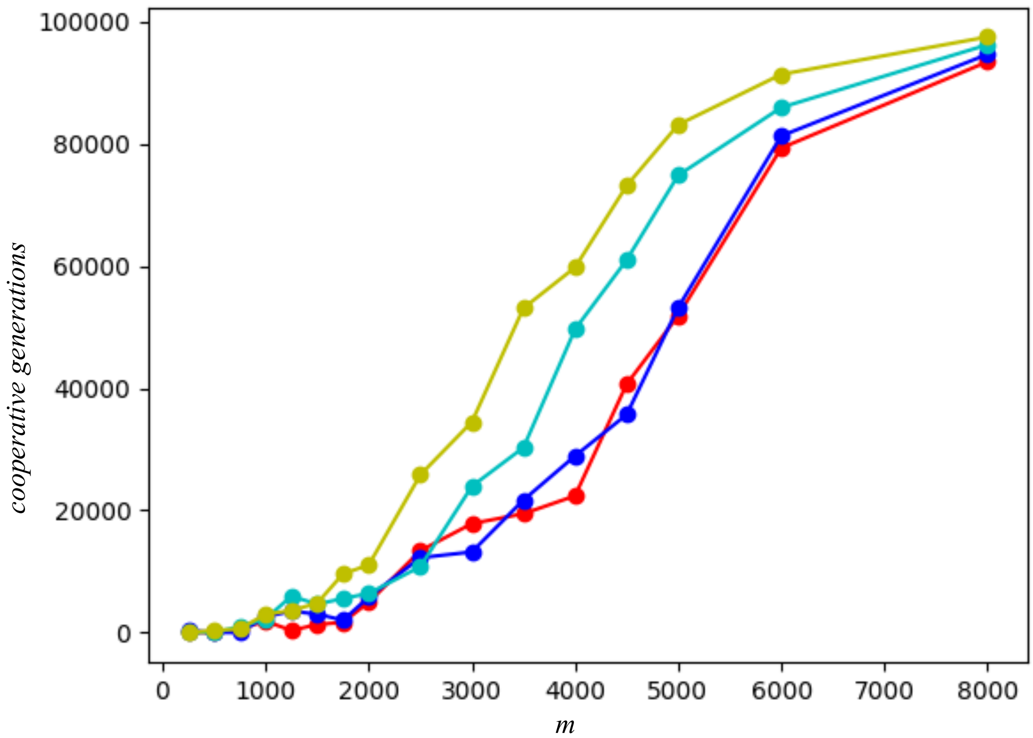


**Figure A6.** The number of generations in which the population was in the cooperative state across varying *m*. Different colors represent the results when all individuals were forced to be non-punishing and without the PRD (red), when the punishing trait was allowed to evolve under the condition that no individuals have the PRD (dark blue), when both punishing and PRD traits were allowed evolve (light blue), and when the punishing trait was allowed to evolve under the condition that all individuals have the PRD (yellow). Parameter values are the same as in Figure 3.